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A $3d$ topological sigma model and D-branes

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ABSTRACT: A 3d topological sigma model describing maps from a 3-manifold Y to a Calabi-Yau 3-fold M is introduced. As the model is topological, we can choose an arbitrary metric on M. Upon scaling up the metric, the path integral by construction localizes on the moduli space of special lagrangean submanifolds of M. We couple the theory to dynamical gauge fields and discuss the case where M has a mirror and the gauge group is U(1).

KEYWORDS: Topological Field Theories, D-branes.

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Contents

1. Introduction

Two dimensional topological sigma models [1] have found many interesting applications both in physics and mathematics. In string theory, for example, the effective field theory of curved D3-branes wrapped around holomorphically embedded 2-cycles in a hyper Kähler manifold, in a limit where the cycle shrinks to zero size, turns out to be a 2d topological sigma model [2]. This effective field theory has then been used to study the existence of bound states of D-branes. From the mathematical point of view, topological sigma models have provided an alternative formulation of Floer groups and Jones polynomials in knot theory. As in the case of Donaldson theory, where physical reformulation [3] has proven to be very useful and enlightening, specially in the analysis and determination of Donaldson invariants [4, 5], one may also hope that the reformulation of Floer groups and Jones polynomials in terms of a topological quantum field theory shed light on the structure of these topological invariants.

The sigma model constructed in [1] (the A-model [6]) is a theory of maps $X : \Sigma \to M$, where Σ is a Riemann surface and M a complex manifold. The topological structure of the model allows one to choose an arbitrary metric on M. When the metric is scaled up, it can be seen that the dominent contributions to the path integral come from the holomorphic maps satisfying

$$\partial_{\bar{z}} X_\alpha = 0\,, \tag{1.1}$$

where $z, \bar{z}$ are complex coordinates on Σ, and $\alpha, \bar{\alpha}$ indicate the complex tangent indices on M.

Interestingly, the holomorphicity condition (1.1) also appears in the study of D-branes wrapping around Riemann surfaces embedded in M. Roughly speaking, type II string theory compactified on $\mathbb{R}^6 \times M$, M being a hyper Kähler manifold, has BPS states in the form of D3-branes wrapping around $S^1 \times \Sigma$. Σ is embedded in M through X, so X is part of the brane coordinates in the ambient space. One can see that these solitonic states in general break the supersymmetry of the underlying superstring theory unless the embedding maps are holomorphic and the U(1) connection (being part of the low energy degrees of freedom) is flat.

Considering D3-branes wrapping around 3-cycles in a Calabi-Yau 3-fold M and demanding supersymmetry leads to some new constraints known as the special lagrangean submanifold conditions [7, 8]

$$X^* k = 0\,, \qquad X^*(\mathrm{Im}\,\Omega) = 0\,. \tag{1.2}$$

Here X is an embedding of a 3-manifold Y to M. k and Ω represent the Kähler and holomorphic 3-form on M, respectively, and $*$ indicates the pull-back operation. The 3-cycles satisfying eqs. (1.2) are sometimes called supersymmetric 3-cycles.

Now taking eqs. (1.2) as the corresponding equations to (1.1) in 3 dimensions, it is interesting to see whether the above 2-dimensional sigma model can be generalized to 3 dimensions. If such, then the path integral of the corresponding sigma model, in some scaling limit of the parameters entering the lagrangean, would localize on the moduli space of solutions to (1.2). As in two dimensions, one may expect that the effective low energy description of curved D5-branes wrapping around $S^1 \times S^1 \times Y$, where Y is a supersymmetric 3-cycle in M shrinking to zero size, is given by a 3d topological sigma model of above type. In [9], using the BV method, a 3 dimensional sigma model has been constructed which has some common properties with the one that we will introduce here. Our motivation, approach, and results, though, are different.

This paper is organized as follows. To begin with, in section 2 we take equations (1.2) as the starting point for the construction of a 3-dimensional topological sigma model. Apart from the bosonic fields X, we introduce some fermionic fields needed to define a BRST-like symmetry. Having had the equations and BRST symmetry, we proceed to construct a lagrangean which is BRST trivial. Next we discuss the moduli space of solutions to (1.2) where the dominent contributions to the path integral come from. In section 3, we couple the theory to dynamical gauge fields. As in [1], a consistent coupling to the gauge fields requires a modification of the BRST transformation rules. The lagrangean describing the dynamics of gauge multiplet is then obtained by dimensional reduction of the lagrangean of twisted $N = 2$ SYM in four dimensions. The case where the gauge group is U(1) and the target manifold M has a mirror $\widetilde{M}$ is of particular interest. The moduli space of solutions to the fixed point equations, in this case, turns out to be parametrized by the mirror manifold $\widetilde{M}$. At the end, we have collected the conventions and some derivations in an appendix.

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2. A topological sigma model

In this section we aim to construct a topological sigma model of maps $X : Y \to M$, which has the special lagrangean conditions (1.2) as its fixed point equations. Let us indicate the indices on Y by $i, j, \ldots$, and those on M by $\mu, \nu, \ldots$ It will prove convenient, if instead of (1.2), we consider the equations $s = \mathsf{k}_{ij} = 0$, with

$$\begin{aligned} s &= 1 + \frac{1}{3!\sqrt{h}}\epsilon^{ijk} X_i^\mu X_j^\nu X_k^\lambda \Omega_{\mu\nu\lambda} \,, \\ \mathsf{k}_{ij} &= X_i^\mu X_j^\nu k_{\mu\nu} \,, \end{aligned}$$

where we have defined $X_i^\mu \equiv \partial_i X^\mu$, and h indicates the determinant of the induced metric;

$$h = \det h_{ij} = \frac{1}{3!}\{\epsilon^{ijk}\epsilon^{mnl} h_{im} h_{jn} h_{kl}\} = \det(\partial_i X^\mu \partial_j X^\nu g_{\mu\nu}) \,.$$

As we are describing the theory on the embedded 3-manifold, in the following, we always use the induced metric to raise or lower the indices on Y. The prescription for the construction of the lagrangean is now as follows.

First we introduce a ghost field ξ^μ, the fermionic partner of X^μ, and the BRST operator δ with an action

$$\delta X^\mu = i\epsilon\, \xi^\mu \,, \qquad \delta\xi^\mu = 0 \,, \tag{2.1}$$

where ϵ is a constant anticommuting parameter. ξ^μ is a section of $X^*(T)$, with T the tangent bundle of M. Further we need to introduce an anti-ghost field χ, an anti-ghost two-form ρ_{ij} (the conjugate fields to s and k_{ij}), as well as a scalar field H and a two-form H_{ij}. The transformation laws are now defined to be

$$\begin{aligned} \delta(h^{1/4}\chi) &= \epsilon\, h^{1/4} H \,, \qquad \delta(h^{1/4} H) = 0 \,, \\ \delta(h^{1/4}\rho_i{}^j) &= \epsilon\, h^{1/4} H_i{}^j \,, \qquad \delta(h^{1/4} H_i{}^j) = 0 \,. \end{aligned} \tag{2.2}$$

Let us define the operator Q by $\delta\Phi = -i\epsilon\{Q, \Phi\}$, for any field Φ. We would like the lagrangean to be a BRST commutator, i.e. $\mathcal{L} = i\{Q, \Psi\}$, for some gauge fermion Ψ. A minimal choice for Ψ is

$$\Psi = \frac{1}{4\lambda}\sqrt{h}\left(\bar{\chi} H + \chi \bar{H} + \rho_{ij} H^{ij} - 2\bar{\chi} s - 2\chi\bar{s} - 2\rho_{ij}\mathsf{k}^{ij}\right) , \tag{2.3}$$

where λ is an arbitrary real parameter. The action now reads

$$\begin{aligned} S = i\int \{Q, \Psi\} = \frac{1}{2\lambda}\int \sqrt{h}\, d^3\sigma \Big[&-H\bar{H} + \bar{H} s + H\bar{s} - \frac{1}{2} H_{ij} H^{ij} + H_{ij}\mathsf{k}^{ij} - \\ &- i h^{-1/4}\bar{\chi}\frac{\delta(h^{1/4} s)}{\delta X^\rho}\xi^\rho - i h^{-1/4}\chi\frac{\delta(h^{1/4}\bar{s})}{\delta X^\rho}\xi^\rho - \\ &- i h^{-1/4}\rho_i{}^j \frac{\delta(h^{1/4} h^{ik}\mathsf{k}_{kj})}{\delta X^\rho}\xi^\rho \Big] \,. \end{aligned}$$

The auxiliary fields H and H_{ij} have no dynamics and can be integrated out using their equations of motion. Doing this, the bosonic part of the action becomes

$$\frac{1}{\lambda}\int \sqrt{h}\, d^3\sigma\, \left(\frac{1}{2}s\bar{s}+\frac{1}{4}\mathsf{k}^{ij}\mathsf{k}_{ij}\right).$$

However, in the appendix we show that

$$\frac{1}{2}s\bar{s}+\frac{1}{4}\mathsf{k}^{ij}\mathsf{k}_{ij} = 1+\frac{1}{2\cdot 3!\sqrt{h}}\epsilon^{ijk}X_i^\mu X_j^\nu X_k^\lambda(\Omega_{\mu\nu\lambda}+\Omega^\dagger_{\mu\nu\lambda})\,. \tag{2.4}$$

Therefore, using the above identity, we can write the action as

$$\begin{aligned} S = \frac{1}{\lambda}\int d^3\sigma\sqrt{h}\Bigg[&1+\frac{1}{2\cdot 3!\sqrt{h}}\epsilon^{ijk}X_i^\mu X_j^\nu X_k^\lambda(\Omega_{\mu\nu\lambda}+\Omega^\dagger_{\mu\nu\lambda}) - \\ &-\frac{i}{4}g_{\mu\nu}h^{ij}X_i^\mu\,\bar{\chi}\nabla_j\xi^\nu\left(1+\frac{1}{3!\sqrt{h}}\epsilon^{mnl}X_m^\rho X_n^\sigma X_l^\delta\Omega_{\rho\sigma\delta}\right)+\text{h.c.} - \\ &-\frac{i}{4\sqrt{h}}\epsilon^{ijk}X_j^\nu X_k^\lambda\Omega_{\mu\nu\lambda}\,\bar{\chi}\partial_i\xi^\mu - \frac{i}{2\cdot 3!\sqrt{h}}\epsilon^{ijk}X_i^\mu X_j^\nu X_k^\lambda\partial_\rho\Omega_{\mu\nu\lambda}\bar{\chi}\xi^\rho+\text{h.c.} - \\ &-\frac{i}{4}h^{ij}g_{\mu\nu}X_i^\mu\mathsf{k}_{mn}\rho^{mn}\nabla_j\xi^\nu + ig_{\mu\nu}\rho^{ik}\mathsf{k}^j{}_k X_i^\mu\nabla_j\xi^\nu - \\ &-ik_{\mu\nu}X_i^\mu\rho^{ij}\partial_j\xi^\nu - \frac{i}{2}\partial_\rho k_{\mu\nu}X_i^\mu X_j^\nu\rho^{ij}\xi^\rho\Bigg]\,, \end{aligned} \tag{2.5}$$

where $\nabla_i\xi^\mu = \partial_i\xi^\mu + \partial_i X^\nu\Gamma^\mu_{\nu\lambda}\xi^\lambda$.

Since the Kähler form is closed, the last two terms combine to

$$\begin{aligned} \rho^{ij}&\left(\partial_i\xi^\mu X_j^\nu k_{\mu\nu}+\partial_j\xi^\nu X_i^\mu k_{\mu\nu}+X_i^\mu X_j^\nu\partial_\rho k_{\mu\nu}\xi^\rho\right) = \\ &= \rho^{ij}\left(\partial_i\xi^\mu X_j^\nu k_{\mu\nu}+\partial_j\xi^\nu X_i^\mu k_{\mu\nu}+\partial_i k_{\rho\nu}X_j^\nu\xi^\rho+\partial_j k_{\mu\rho}X_i^\mu\xi^\rho\right) \\ &= \rho^{ij}\left(\partial_i(X_j^\nu\xi^\mu k_{\mu\nu})-\partial_j(X_i^\nu\xi^\mu k_{\mu\nu})\right)\,. \end{aligned} \tag{2.6}$$

Likewise, as the holomorphic 3-form is closed, the terms in the third line can be written as

$$\begin{aligned} &\frac{1}{2\sqrt{h}}\epsilon^{ijk}\bar{\chi}\left\{\partial_i\xi^\mu X_j^\nu X_k^\lambda\Omega_{\mu\nu\lambda}+\frac{1}{3}X_i^\mu X_j^\nu X_k^\lambda\partial_\rho\Omega_{\mu\nu\lambda}\xi^\rho\right\} = \\ &= \frac{1}{3!\sqrt{h}}\epsilon^{ijk}\bar{\chi}\left\{\partial_{[i}\xi^\mu X_j^\nu X_{k]}^\lambda\Omega_{\mu\nu\lambda}+\xi^\rho\left(X_j^\nu X_k^\lambda\partial_i\Omega_{\rho\nu\lambda}+X_i^\mu X_k^\lambda\partial_j\Omega_{\mu\rho\lambda}+X_i^\mu X_j^\nu\partial_k\Omega_{\mu\nu\rho}\right)\right\} \\ &= \frac{1}{2\sqrt{h}}\epsilon^{ijk}\bar{\chi}\,\partial_i(\xi^\mu X_j^\nu X_k^\lambda\Omega_{\mu\nu\lambda})\,. \end{aligned} \tag{2.7}$$

Let $\chi = \varrho + i\zeta$, where ϱ and ζ are real fields, the action now reads

$$
\begin{aligned}
S = \frac{1}{\lambda}\int d^3\sigma\sqrt{h}\Big[1 &+ \frac{1}{2\cdot 3!\sqrt{h}}\epsilon^{ijk}X_i^\mu X_j^\nu X_k^\lambda(\Omega_{\mu\nu\lambda}+\Omega^\dagger_{\mu\nu\lambda}) - \\
&-\frac{i}{2}\left(1+\frac{1}{2\cdot 3!\sqrt{h}}\epsilon^{mnl}X_m^\rho X_n^\sigma X_l^\delta(\Omega_{\rho\sigma\delta}+\Omega^\dagger_{\rho\sigma\delta})\right) g_{\mu\nu}h^{ij}X_i^\mu\,\varrho\nabla_j\xi^\nu - \\
&-\frac{1}{4\cdot 3!\sqrt{h}}\epsilon^{mnl}X_m^\rho X_n^\sigma X_l^\delta(\Omega_{\rho\sigma\delta}-\Omega^\dagger_{\rho\sigma\delta})g_{\mu\nu}h^{ij}X_i^\mu\,\zeta\nabla_j\xi^\nu - \\
&-\frac{i}{4\sqrt{h}}\epsilon^{ijk}\left(\varrho\,\partial_i\left(\xi^\mu X_j^\nu X_k^\lambda(\Omega_{\mu\nu\lambda}+\Omega^\dagger_{\mu\nu\lambda})\right) - i\zeta\,\partial_i\left(\xi^\mu X_j^\nu X_k^\lambda(\Omega_{\mu\nu\lambda}-\Omega^\dagger_{\mu\nu\lambda})\right)\right) - \\
&-\frac{i}{4}h^{ij}g_{\mu\nu}X_i^\mu\mathsf{k}_{mn}\rho^{mn}\nabla_j\xi^\nu + ig_{\mu\nu}\rho^{ik}\mathsf{k}^j{}_k X_i^\mu\nabla_j\xi^\nu - i\rho^{ij}\partial_i(X_j^\nu\xi^\mu k_{\mu\nu})\Big]\,. \qquad (2.8)
\end{aligned}
$$

Recall that the action was defined to be a BRST commutator. Therefore the path integral – upon assuming that the measure is also invariant under the BRST transformations – is independent of the parameter λ as well as the metric on M (or more generally independent of any data entering Ψ). Hence, in computing the path integral, an arbitrary and convenient value for these parameters can be chosen. For instance, we may take the limit $\lambda \to 0$. Or equivalently, we may scale the metric as $g_{\mu\nu} \to tg_{\mu\nu}$ for t a real parameter, and then take the limit $t \to \infty$. In both limits, the path integral will localize around the solutions of the following equations

$$\sqrt{h} + \frac{1}{2\cdot 3!}\epsilon^{ijk}X_i^\mu X_j^\nu X_k^\lambda(\Omega_{\mu\nu\lambda}+\Omega^\dagger_{\mu\nu\lambda}) = 0\,.$$

By eq. (2.4), this is equivalent to eqs. (1.2) defining the special lagrangean submanifolds of M. Let us indicate the moduli space of solutions to these equations by $\mathcal{M}_{sl}$ and study the tangent space at Y.

2.1 The tangent space to $\mathcal{M}_{sl}$

Let X^μ describe a special lagrangean submanifold of M. We deform X^μ to a nearby map $X^\mu + \delta X^\mu$, and ask under what conditions this new map is still a special lagrangean submanifold. For δX^μ not to have any component obtainable by a diffeomorphism on Y, we demand that δX^μ belong to the normal bundle at Y, i.e.

$$g_{\mu\nu}\partial_i X^\mu \delta X^\nu = 0\,.$$

To find the tangent space to $\mathcal{M}_{sl}$ at Y, we impose the special lagrangean conditions ($\mathrm{Im}\, s = \mathsf{k}_{ij} = 0$) on $X^\mu + \delta X^\mu$. Following the same line of arguments as in (2.6), for the lagrangean condition (i.e. $\mathsf{k}_{ij} = 0$) we get

$$0 = \partial_{[i}(\delta X^\mu)X_{j]}^\nu k_{\mu\nu} = \partial_i(X_j^\nu\delta X^\mu k_{\mu\nu}) - \partial_j(X_i^\nu\delta X^\mu k_{\mu\nu}) \equiv \partial_i\omega_j - \partial_j\omega_i\,, \qquad (2.9)$$

where $\omega_i \equiv \delta X^\mu X_i^\nu k_{\mu\nu}$. As the pull-back of the Kähelr form is zero, the complex structure provides an isomorphism between the tangent space, $T_Y M$, and the normal space, $N_Y M$ at Y. As such, $J^\mu{}_\nu \delta X^\nu$ is a tangent vector at Y. Contracting this vector with the metric and pulling it back gives a one-form on Y. Hence, ω_i is a one-form on Y.

The implication of the special lagrangean condition can be seen more easily by writing the Kähler form and Ω in the vielbein bases (here $n = 3$),

$$
\begin{aligned}
k &= E^1 \wedge E^{n+1} + E^2 \wedge E^{n+2} + E^3 \wedge E^{n+3} \,, \\
\Omega &= (E^1 + iE^{n+1}) \wedge (E^2 + iE^{n+2}) \wedge (E^3 + iE^{n+3}) \,,
\end{aligned}
$$

or in components

$$
\begin{aligned}
k_{\mu\nu} &= E^a_\mu E^{n+a}_\nu - E^a_\nu E^{n+a}_\mu \,, \\
\Omega_{\mu\nu\lambda} &= \epsilon_{abc}(E^a + iE^{n+a})_\mu (E^b + iE^{n+b})_\nu (E^c + iE^{n+c})_\lambda \,,
\end{aligned}
\tag{2.10}
$$

where $a, b, \ldots = 1, 2, 3$. We adopt the vielbeins such that E^a_μ and E^{n+a}_μ span the tangent and normal spaces at Y, respectively. The pull-back of E^a_μ is then naturally to be the vielbeins e^a_i on Y

$$
e^a_i = X^\mu_i E^a_\mu \,, \qquad X^\mu_i E^{a+n}_\mu = 0 \,, \qquad \delta X^\mu E^a_\mu = 0 \,. \tag{2.11}
$$

If we impose the second condition $\mathrm{Im}\, s = \frac{1}{3!\sqrt{h}} \epsilon^{ijk} X^\mu_i X^\nu_j X^\lambda_k \mathrm{Im}\, \Omega_{\mu\nu\lambda} = 0$ on $X^\mu + \delta X^\mu$ we get

$$
\begin{aligned}
0 &= \epsilon^{ijk} \mathrm{Im} \left\{ \partial_i (\delta X^\mu) X^\nu_j X^\lambda_k \Omega_{\mu\nu\lambda} + \frac{1}{3} X^\mu_i X^\nu_j X^\lambda_k \partial_\rho \Omega_{\mu\nu\lambda} \delta X^\rho \right\} \\
&= \epsilon^{ijk} \mathrm{Im}\, \partial_i (\delta X^\mu X^\nu_j X^\lambda_k \Omega_{\mu\nu\lambda}) \,.
\end{aligned}
\tag{2.12}
$$

using (2.10) and (2.11), this can be written as

$$
\begin{aligned}
0 &= \partial_i (\epsilon^{ijk} \epsilon_{abc} \delta X^\mu E^{n+a}_\mu X^\nu_j E^b_\nu X^\lambda_k E^c_\lambda) \\
&= \partial_i (\epsilon^{ijk} \epsilon_{abc} \delta X^\mu E^{n+a}_\mu e^b_j e^c_k) \\
&= \partial_i (\epsilon^{ijk} \epsilon_{ljk} \delta X^\mu E^{n+a}_\mu e^l_a) \\
&= 2\partial_i (\delta X^\mu E^{n+a}_\mu e^i_a) \\
&= 2\partial_i (\delta X^\mu E^{n+a}_\mu E^\nu_a X^i_\nu) \\
&= 2\partial_i (\delta X^\mu X^{i\nu} k_{\mu\nu}) = 2\partial_i \omega^i \,.
\end{aligned}
\tag{2.13}
$$

Equations (2.9) and (2.13) imply that for the map $X^\mu + \delta X^\mu$ to be a special lagrangean submanifold of M, the one-form $\omega_i = \delta X^\mu X^\nu_i k_{\mu\nu}$ must be closed and coclosed [10, 11]. Therefore any harmonic one-form ω on Y specifies a direction in which a special lagrangean submanifold can be deformed in $\mathcal{M}_{sl}$, and the dimension of $\mathcal{M}_{sl}$ equals b_1, the first Betti number of the manifold Y. Looking back to (2.8) shows that, in the background of a map X^μ describing a special lagrangean submanifold of M, zero modes of ξ^μ satisfy the same equations that δX^μ does in (2.9) and (2.13). Consequently, if $b_1 > 0$, there exist ξ^μ zero modes and the partition function identically vanishes. To soak up the zero modes, while maintaining the topological characteristic of the theory, we need to insert some BRST invariant operators with the right ghost number in to the path integral. This leads us to look for the cohomology classes of the operator Q. Following [1], these cohomology classes can be constructed and be shown that are in correspondence with the de Rham cohomology classes of M.

3. Coupling to gauge fields

We would like now to couple the sigma model of the previous section to dynamical gauge fields of a compact gauge group G. As in [1], G is taken to be the group of automorphisms of M preserving all the structures on the manifold. The G action on M is through the vector fields $V_a = V_a^\mu e_\mu$, $a = 1, \ldots, n$ (n is the dimension of G), representing the group generators, i.e.

$$[V_a, V_b] = f_{ab}{}^c V_c \, ,$$

for f_{abc} the structure constants of G. The fact that the G action preserves all the structures on M means that the Lie derivatives of the metric, Kähler form, and holomorphic 3-form all have to vanish:

$$\mathcal{L}_{V_a}(g) = \mathcal{L}_{V_a}(k) = \mathcal{L}_{V_a}(\Omega) = 0 \, ,$$

implying that

$$\begin{aligned} \nabla_\mu V_{a\nu} + \nabla_\nu V_{a\mu} &= 0 \\ k_{\rho\nu} \nabla_\mu V_a^\rho + k_{\mu\rho} \nabla_\nu V_a^\rho &= 0 \\ \Omega_{\mu\nu\rho} \nabla_\lambda V_a^\rho + \Omega_{\mu\rho\lambda} \nabla_\nu V_a^\rho + \Omega_{\rho\nu\lambda} \nabla_\mu V_a^\rho &= 0 \, . \end{aligned}$$

In the following, first we discuss the lagrangean of the gauge sector. Next, as the gauge group acts nontrivially on M and since δ^2 acting on the gauge multiplet produces a gauge transformation, we will see how the transformation rules in (2.1) need to be modified.

The topological lagrangean describing the dynamics of gauge sector can be obtained more conveniently by dimensional reduction of Donaldson-Witten theory in four dimensions.[1] The lagrangean of the Donaldson-Witten theory, on the other hand, can be obtained by twisting the lagrangean of $N = 2$ SYM theory on $\mathbb{R}^4$. As a result, one finds the following field content and lagrangean. In the bosonic sector there are a gauge field A_μ, two scalars ϕ and $\bar{\phi}$, and in the fermionic sector we

[1] Alternatively, one may follow [12] to construct the lagrangean of the gauge multiplet.

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have a one-form ψ_μ, a scalar η, and a self-dual two-form $\chi_{\mu\nu}$. The lagrangean on an arbitrary smooth four manifold then reads [3]

$$\mathcal{L} = \mathrm{tr}\left[\frac{1}{4}F^+_{\mu\nu}F^{+\mu\nu} + \frac{1}{2}\phi D_\mu D^\mu\bar{\phi} - i\eta D_\mu\psi^\mu + iD_\mu\psi_\nu\chi^{\mu\nu} - \right.$$
$$\left. - \frac{i}{8}\phi[\chi_{\mu\nu},\chi^{\mu\nu}] - \frac{i}{2}\bar{\phi}[\psi_\mu,\psi^\mu] - \frac{i}{2}\phi[\eta,\eta] - \frac{1}{8}[\phi,\bar{\phi}]^2\right]. \tag{3.1}$$

This action is invariant under the transformations

$$\begin{aligned} \delta A_\mu &= i\epsilon\,\psi_\mu\,, & \delta\phi &= 0\,, & \delta\bar{\phi} &= 2i\epsilon\,\eta\,,\\ \delta\eta &= \frac{1}{2}\epsilon[\phi,\bar{\phi}]\,, & \delta\psi_\mu &= -\epsilon\,D_\mu\phi\,, & \delta\chi_{\mu\nu} &= 2\epsilon\,F^+_{\mu\nu}\,. \end{aligned} \tag{3.2}$$

Take the underlying four manifold to be $Y\times\mathbb{R}$, for Y the embedded 3-manifold. Let t parametrize the line $\mathbb{R}$ and simply require that the fields not depend on t. Setting $\varphi = A_0$, $a = \psi_0$, $\chi_i = \chi_{i0} = \frac{1}{2}\epsilon_{ijk}\chi^{jk}$, the lagrangean describing the dynamics of gauge multiplet on Y reads

$$\mathcal{L}_G = \mathrm{tr}\left[\frac{1}{4}F_{ij}F^{ij} + \frac{1}{2}\epsilon_{ijk}F^{ij}D^k\varphi - \frac{1}{2}D_i\varphi D^i\varphi + \frac{1}{2}\phi D_iD^i\bar{\phi} - i\eta D_i\psi^i - \right.$$
$$- i\chi_iD^ia - i\epsilon_{ijk}\chi^iD^j\psi^k - [\varphi,\psi_i]\chi^i - \frac{i}{2}\phi[\chi_i,\chi^i] - \frac{i}{2}\bar{\phi}[\psi_i,\psi^i] -$$
$$\left. - \frac{i}{2}\bar{\phi}[a,a] + \eta[\varphi,a] + \frac{1}{2}[\varphi,\phi][\varphi,\bar{\phi}] - \frac{i}{2}\phi[\eta,\eta] - \frac{1}{8}[\phi,\bar{\phi}]^2\right]\,. \tag{3.3}$$

The symmetry transformations of this action are obviously read from (3.2) to be

$$\begin{aligned} \delta A_i &= i\epsilon\,\psi_i\,, & \delta\bar{\phi} &= 2i\epsilon\,\eta\,, & \delta\eta &= \frac{1}{2}\epsilon[\phi,\bar{\phi}]\,,\\ \delta\varphi &= i\epsilon\,a\,, & \delta a &= -\epsilon[\varphi,\phi]\,, & \delta\phi &= 0\,,\\ \delta\psi_i &= -\epsilon\,D_i\phi\,, & \delta\chi_i &= \epsilon\left(\frac{1}{2}\epsilon_{ijk}F^{jk} + D_i\varphi\right)\,. \end{aligned} \tag{3.4}$$

As Y has no boundary, the second term in the action vanishes by the Bianchi identity. So the fixed point equations reduce to

$$F_{ij} = 0\,, \qquad D_i\phi = D_i\varphi = 0\,.$$

Notice that $\delta_G \equiv \delta^2$ acting on the fields in (3.4) produces an infinitesimal gauge transformation. So to couple the gauge multiplet to the sigma model we need to change the transformation rules in (2.1) such that δ^2 is not zero but a gauge transformation. Firstly, note that the infinitesimal action of G on the coordinates is $\delta_G X^\mu \sim \phi^a V^\mu_a$, for ϕ^a the gauge parameter. So to have $\delta^2 = \delta_G$, we need to change the transformations (2.1) to

$$\delta X^\mu = i\epsilon\,\xi^\mu\,, \qquad \delta\xi^\mu = \epsilon\,\phi^a V^\mu_a\,.$$

This, in particular, gives the proper gauge transformation of ξ^μ as a section of $X^*(T)$:

$$\delta_G \xi^\mu \sim \phi^a \partial_\nu V_a^\mu \xi^\nu \,.$$

With this change, the covariant derivative of X^μ, $D_i X^\mu = \partial_i X^\mu + A_i^a V_a^\mu$, transforms (just like ξ^μ) as a vector under δ_G. As we are interested in having a set of gauge invariant equations, and since the metric, k and Ω are all invariant under the G action, we replace s and k_{ij} with

$$\begin{aligned} s &= 1 + \frac{1}{3!}(\det(g_{\mu\nu} D_i X^\mu D_j X^\nu))^{-1/2} \epsilon^{ijk} D_i X^\mu D_j X^\nu D_k X^\lambda \Omega_{\mu\nu\lambda} \,, \\ \mathsf{k}_{ij} &= D_i X^\mu D_j X^\nu k_{\mu\nu} \,. \end{aligned}$$

Let us now see if any changes need to be made in the transformation rules of χ and ρ_{ij}. Since the action is BRST-exact and we want to maintain its invariance under δ, Ψ has to be a gauge invariant quantity

$$0 = \delta S = \delta^2 \Psi = \delta_G \Psi \,.$$

On the other hand, as s and k are gauge invariant, a look back to (2.3) shows that the conjugate fields χ and ρ_{ij} also have to be gauge invariant. This implies that the transformation rules (2.2) do not need any changes. The gauge invariant version of the sigma model action can now be derived by varying the Ψ with respect to the newly defined δ, $\mathcal{L}_{\mathrm{sig}} = i\{Q, \Psi\}$. This turns out to be the action in (2.5), if we do the following substituations:

$$\begin{aligned} \partial_i X^\mu &\to D_i X^\mu \,, \\ \partial_i \xi^\mu &\to \partial_i \xi^\mu + A_i^a \partial_\nu V_a^\mu \xi^\nu \,. \end{aligned} \tag{3.5}$$

The total lagrangean is then the sum of the lagrangeans $\mathcal{L}_{\mathrm{sig}}$ and (3.3)

$$\mathcal{L} = \mathcal{L}_G + \mathcal{L}_{\mathrm{sig}} \,.$$

3.1 The U(1) case

In the case of U(1) gauge group with a trivial action on M, the fixed point equations constrain the map X^μ to be a special lagrangean submanifold, and the U(1) gauge connection living on the embedded 3-manifold to be flat. In the previous section, we saw that $\mathcal{M}_{sl}$ is parametrized by a torus T^{b_1}. It is also easy to see that the moduli space of flat U(1) connections on Y is parametrized by a torus T^{b_1}. So the dimension of the total moduli space $\mathcal{M}$ is $2b_1$. Suppose M has a mirror $\widetilde{M}$. Quantum mirror symmetry then implies that the total moduli space $\mathcal{M}$ is nothing but the mirror manifold $\widetilde{M}$, and $b_1 = 3$ for the dimensions to match [11].

We conclude that for the U(1) gauge group, the path integral calculation of the corresponding topological sigma model reduces to an integral over the mirror manifold. The lagrangean of such a topological sigma model coupled to gauge fields should in principle be obtainable starting from the lagrangean of super Yang-Mills theory on a Calabi-Yau 3-fold [13] and then reducing it on to a $3d$ submanifold. We hope to return to this point in future.

A. Conventions and the proof of eq. (2.4)

In this appendix, using the conventions of [13], we derive equation (2.4).

Let M be a Calabi-Yau 3-fold. As such, there exists a spinor θ on M which is a singlet under the holonomy group SU(3). Take θ to be left-handed and normalize it such that $\theta^\dagger\theta = 1$. Let us choose a representation for the gamma matrices such that they are hermitian and antisymmetric; $\gamma_\mu^\dagger = \gamma_\mu$, $\gamma_{\mu\nu}^\dagger = -\gamma_{\mu\nu}$, $\gamma_{\mu\nu\lambda}^\dagger = -\gamma_{\mu\nu\lambda}$ (where $\gamma_{\mu\nu\lambda} = \frac{1}{2}\{\gamma_\mu, \gamma_{\nu\lambda}\}$). Thus

$$\begin{aligned} \gamma_7\theta &= -\theta\,, & \gamma_7\theta^* &= \theta^* \\ \theta^\dagger\gamma_7 &= -\theta^\dagger\,, & \theta^t\gamma_7 &= \theta^t\,, \end{aligned} \tag{A.1}$$

θ and $\theta^\dagger$ are left-handed while θ^* and θ^t are right-handed spinors. This in particular implies that

$$\theta^\dagger\theta^* = \theta^\dagger\gamma_{\mu\nu\lambda}\theta = 0\,.$$

Also, it is easy to see that $\theta^\dagger\gamma_\mu\theta^* = 0$. We define

$$\begin{aligned} k_{\mu\nu} &= i\theta^\dagger\gamma_{\mu\nu}\theta \\ \Omega_{\mu\nu\lambda} &= \theta^\dagger\gamma_{\mu\nu\lambda}\theta^*\,. \end{aligned} \tag{A.2}$$

As θ is a singlet under the holonomy group, these are nowhere vanishing closed forms. Therefore we recognize k and Ω as the Kähler and holomorphic 3-form on M, respectively.

To derive eq. (2.4), first note that

$$\begin{aligned} \Omega_{\mu\nu\lambda}\Omega^\dagger_{\rho\sigma\delta} &= -(\theta^\dagger\gamma_{\mu\nu\lambda}\theta^*)(\theta^t\gamma_{\rho\sigma\delta}\theta) \\ &= -(\theta^\dagger\gamma_{\mu\nu\lambda})\left[\frac{1}{2}(1+\gamma_7) - \frac{1}{2}\gamma^\eta\theta\theta^\dagger\gamma_\eta\right](\gamma_{\rho\sigma\delta}\theta) \\ &= -\theta^\dagger\gamma_{\mu\nu\lambda}\gamma_{\rho\sigma\delta}\theta + \frac{1}{2}(\theta^\dagger\gamma_{\mu\nu\lambda}\gamma^\eta\theta)(\theta^\dagger\gamma_\eta\gamma_{\rho\sigma\delta}\theta)\,, \end{aligned} \tag{A.3}$$

where use has been made of the Fierz identity

$$\theta^*\theta^t + \frac{1}{2}\gamma^\lambda\theta\theta^\dagger\gamma_\lambda = \frac{1}{2}(1+\gamma_7)\,.$$

Using the identity

$$\gamma_{\rho\sigma\delta}\gamma^{\mu\nu\lambda} = \frac{i}{2}\epsilon^{\mu\nu\lambda}{}_{\rho\alpha\beta}\left(-\frac{i}{2}\epsilon^{\alpha\beta}{}_{\sigma\delta\alpha'\beta'}\gamma^{\alpha'\beta'} - \delta_{[\sigma}{}^{[\alpha}\gamma_{\delta]}{}^{\beta]}\gamma_7 - \delta_{[\delta}{}^{[\beta}\delta_{\sigma]}{}^{\alpha]}\gamma_7\right) +$$
$$+\delta_\rho{}^\mu\left(-\frac{i}{2}\epsilon^{\nu\lambda}{}_{\sigma\delta\alpha'\beta'}\gamma^{\alpha'\beta'}\gamma_7 - \delta_{[\sigma}{}^{[\nu}\gamma_{\delta]}{}^{\lambda]} - \delta_{[\delta}{}^{[\lambda}\delta_{\sigma]}{}^{\nu]}\right) +$$
$$+\delta_\rho{}^\lambda\left(-\frac{i}{2}\epsilon^{\mu\nu}{}_{\sigma\delta\alpha'\beta'}\gamma^{\alpha'\beta'}\gamma_7 - \delta_{[\sigma}{}^{[\mu}\gamma_{\delta]}{}^{\nu]} - \delta_{[\delta}{}^{[\nu}\delta_{\sigma]}{}^{\mu]}\right) +$$
$$+\delta_\rho{}^\nu\left(-\frac{i}{2}\epsilon^{\lambda\mu}{}_{\sigma\delta\alpha'\beta'}\gamma^{\alpha'\beta'}\gamma_7 - \delta_{[\sigma}{}^{[\lambda}\gamma_{\delta]}{}^{\mu]} - \delta_{[\delta}{}^{[\mu}\delta_{\sigma]}{}^{\lambda]}\right) +$$
$$+g_{\sigma\rho}\left(\frac{i}{2}\epsilon^{\mu\nu\lambda}{}_{\delta\alpha'\beta'}\gamma^{\alpha'\beta'}\gamma_7 + \delta_\delta{}^\mu\gamma^{\nu\lambda} + \delta_\delta{}^\lambda\gamma^{\mu\nu} + \delta_\delta{}^\nu\gamma^{\lambda\mu}\right) -$$
$$-g_{\delta\rho}\left(\frac{i}{2}\epsilon^{\mu\nu\lambda}{}_{\sigma\alpha'\beta'}\gamma^{\alpha'\beta'}\gamma_7 + \delta_\sigma{}^\mu\gamma^{\nu\lambda} + \delta_\sigma{}^\lambda\gamma^{\mu\nu} + \delta_\sigma{}^\nu\gamma^{\lambda\mu}\right), \qquad \text{(A.4)}$$

the first term in (A.3) can be expanded in terms of the Kähler form. Pulling back this term to Y results in

$$-\frac{1}{(3!)^2 h}\epsilon^{ijk}\epsilon^{mnl}X_i^\mu X_j^\nu X_k^\lambda X_m^\rho X_n^\sigma X_l^\delta(\theta^\dagger\gamma_{\mu\nu\lambda}\gamma_{\rho\sigma\delta}\theta) = 1\,. \qquad \text{(A.5)}$$

Similarly, using

$$\gamma_\rho\gamma_{\mu\nu\lambda} = \frac{i}{2}\epsilon_{\rho\mu\nu\lambda\eta\delta}\gamma^{\eta\delta}\gamma_7 + g_{\rho\mu}\gamma_{\nu\lambda} + g_{\lambda\rho}\gamma_{\mu\nu} + g_{\rho\nu}\gamma_{\lambda\mu}\,,$$

and

$$\epsilon_{\rho\mu\nu\lambda\eta\delta}\,k^{\eta\delta} = 2(k_{\rho\mu}k_{\nu\lambda} + k_{\rho\nu}k_{\lambda\mu} + k_{\mu\nu}k_{\rho\lambda})\,,$$

the second term in (A.3) can be expanded in terms of the Kähler form

$$\frac{1}{2}(\theta^\dagger\gamma_{\mu\nu\lambda}\gamma^\eta\theta)(\theta^\dagger\gamma_\eta\gamma_{\rho\sigma\delta}\theta) =$$
$$= k_{\lambda\nu}(g_{\mu\rho}k_{\sigma\delta} + g_{\mu\delta}k_{\rho\sigma} + g_{\mu\sigma}k_{\delta\rho} + ik_{\rho\mu}k_{\sigma\delta} + ik_{\delta\mu}k_{\rho\sigma} + ik_{\mu\sigma}k_{\rho\delta}) +$$
$$+k_{\nu\mu}(g_{\rho\lambda}k_{\sigma\delta} + g_{\lambda\delta}k_{\rho\sigma} + g_{\lambda\sigma}k_{\delta\rho} + ik_{\rho\lambda}k_{\sigma\delta} + ik_{\delta\lambda}k_{\rho\sigma} + ik_{\delta\rho}k_{\sigma\lambda}) +$$
$$+k_{\mu\lambda}(g_{\rho\nu}k_{\sigma\delta} + g_{\nu\delta}k_{\rho\sigma} + g_{\sigma\nu}k_{\delta\rho} + ik_{\rho\nu}k_{\sigma\delta} + ik_{\delta\nu}k_{\rho\sigma} + ik_{\sigma\nu}k_{\delta\rho})\,.$$

It is now easy to see that

$$\frac{1}{2\cdot(3!)^2 h}\epsilon^{ijk}\epsilon^{mnl}X_i^\mu X_j^\nu X_k^\lambda X_m^\rho X_n^\sigma X_l^\delta(\theta^\dagger\gamma_{\mu\nu\lambda}\gamma^\eta\theta)(\theta^\dagger\gamma_\eta\gamma_{\rho\sigma\delta}\theta) = -\frac{1}{2}\mathsf{k}_{ij}\mathsf{k}^{ij}\,.$$

So finally we can write

$$\frac{1}{2}s\bar{s} = \frac{1}{2}\left(1 + \frac{1}{3!\sqrt{h}}\epsilon^{ijk}X_i^\mu X_j^\nu X_k^\lambda\Omega_{\mu\nu\lambda}\right)\left(1 + \frac{1}{3!\sqrt{h}}\epsilon^{mnl}X_m^\rho X_n^\sigma X_l^\delta\Omega^\dagger_{\rho\sigma\delta}\right)$$
$$= 1 + \frac{1}{2\cdot 3!\sqrt{h}}\epsilon^{ijk}X_i^\mu X_j^\nu X_k^\lambda(\Omega_{\mu\nu\lambda} + \Omega^\dagger_{\mu\nu\lambda}) -$$
$$-\frac{1}{4}(X_i^\mu X_j^\nu k_{\mu\nu})(X_\rho^i X_\sigma^j k^{\rho\sigma})\,,$$

which is equation (2.4). Note that this equation puts a lower bound on the induced volume of Y

$$\int \sqrt{h}\, d^3\sigma \geq -\frac{1}{3!}\int X^*(\mathrm{Re}\,\Omega)\,,$$

and indicates that this bound is saturated if and only if $s = \mathsf{k}_{ij} = 0$.

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